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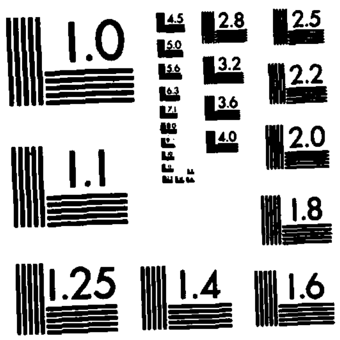

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ADAPTIVE FIXED INTERVAL TRAJECTORY SMOOTHING

TECHNICAL REPORT NO. 85

JANUARY 1983

MATHEMATICAL SERVICES BRANCH
DATA SCIENCES DIVISION
US ARMY WHITE SANDS MISSILE RANGE
WHITE SANDS MISSILE RANGE, NEW MEXICO

ADAPTIVE FIXED INTERVAL TRAJECTORY SMOOTHING

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TECHNICAL REPORT NO. 85

JANUARY 1983

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An adaptive, fixed interval smoother for trajectory estimation is described. The fixed interval smoother uses the modified Bryson-Fraser formulation which combines the state estimates of a forward running Kalman filter and a backward running adjoint filter. The square root formulation used for the Kalman filter is described. The procedures used to adapt the Kalman filter to the local noise content of the measurements and to adapt the filter to abrupt changes in target acceleration are discussed.

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INTRODUCTION

The adaptive, fixed interval smoother uses raw cartesian data input, $x(t_i)$, $y(t_i)$, $z(t_i)$ for $t_i = t_1, t_2, ---, t_f$, where t_1 is the first measurement time and t_f is the final trajectory measurement time, to produce smooth estimates of the trajectory states, position, velocity, and acceleration. In contrast to other smoothers the fixed interval smoother uses the measurements at all trajectory measurement times to estimate the trajectory states at each measurement time. Thus, in a sense one could say that the smoothing interval is the total trajectory time. The fixed interval smoother is based on the modified Bryson-Frazier formulation, denoted by mBF, which was developed by Bierman in [1]. The mBF development of the fixed interval smoother combines a forward running Kalman filter with a backward adjoint filter which uses the Kalman filter residuals as input and also uses other Kalman filter computed quantities. The mBF smoother is a very stable, computationally efficient form of the fixed interval smoother.

The fixed interval smoother is made adaptive by adapting the forward Kalman filter to the local noise content of the raw position data and also adapting the filter to acceleration changes which are sensed in the future measurements. The Kalman filter is initialized with least squares state estimates at the desired start time and again after any large time break in the measurement sequence. The smoothing program expects that the cartesian measurement input has been preprocessed to the extent that it is free from wild observations. The smoother outputs trajectory position, velocity, and acceleration and estimates of the errors associated with these quantities. The trajectory states and their error estimates can be further processed by an output routine to rotate and translate the states to a desired coordinate system and origin, to compute quantities derivable from cartesian position, velocity, and acceleration, to combine the trajectory states with atmospheric measurements, and to reformat the output.

TRAJECTORY AND MEASUREMENT MODELS

Let $s(t_k)$ be a 3-vector which represents any of the three coordinates of the trajectory, i.e., $s = (x, \dot{x}, \ddot{x})$, or $s = (y, \dot{y}, \ddot{y})$, or $s = (z, \dot{z}, \ddot{z})$. Assume that each coordinate of the trajectory obeys the discrete time dynamic state equation

$$s(t_{k+1}) = \Phi(\Delta_k)\, s(t_k) + \gamma(\Delta_k)u, \tag{1}$$

where $\Delta_k = t_{k+1} - t_k$. $\Phi(\Delta_k)$ is the second order transition matrix

$$\Phi(\Delta_k) = \begin{bmatrix} 1 & \Delta_k & \Delta_k^2 \\ 0 & 1 & \Delta_k \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

u is an unknown, constant scalar forcing function and $\gamma(\Delta_k)$ is the vector

$$\gamma^T(\Delta_k) = [\ \Delta_k^3/6 \quad \Delta^2{}_k/2 \quad \Delta_k\ ] \tag{3}$$

Let $m(t_i)$ denote the position measurement available to the smoother. $m(t_i)$ is represented as

$$m(t_i) = Hs(t_i) + e(t_i), \tag{4}$$

where $H = [100]$ and $e(t_i)$ is a zero mean measurement error with variance $R(t_i)$.

KALMAN FILTER

Let $\hat{s}(k/k)$ and $\hat{s}(k/k-1)$ denote the filtered and predicted state estimates at time t_k. Also, let $P(k/k)$ and $P(k/k-1)$ denote the covariance matrices of the filtered and predicted state estimates at time t_k. The state estimates and their covariances are determined by the usual Kalman filter equations,

$$\hat{s}(k+1/k) = \Phi(\Delta_k)\hat{s}(k/k) \tag{5}$$

$$P(k+1/k) = \Phi(\Delta_k)P(k/k)\Phi^T(\Delta_k) + q(k)\gamma(\Delta_k)\gamma^T(\Delta_k) \tag{6}$$

$$P(k+1/k+1) = P(k+1/k) - P(k+1/k)H^T(HP(k+1/k)H^T + R(t_{k+1}))^{-1}HP(k+1/k) \tag{7}$$

$$\hat{s}(k+1/k+1) = \hat{s}(k+1/k) + P(k+1/k+1)H^TR^{-1}(t_{k+1})(m(t_{k+1}) - H\hat{s}(k+1/k)) \tag{8}$$

The filter equations given in (5) - (8) are not implemented directly, but are implemented in a square root form. Let the covariance matrix $P(k/k)$ be represented as $P(k/k) = U(k/k)D(k/k)U^T(k/k)$ where $U(k/k)$ is unit upper triangular and $D(k/k)$ is diagonal with positive diagonal elements. Also, $P(k+1/k) = U(k+1/k)D(k+1/k)U^T(k+1/k)$. The measurement noise variance, $R(t_k)$, in the filter equations represents the noise variance in the vicinity of time t_k. $q(k)$ is a scalar representing the uncertainty of the unknown forcing term, u.

At a time update the upper triangular factor U and diagonal factor D of the predicted covariance matrix are updated rather than computing an updated covariance, $P(k+1/k)$. The updated U-D factors $U(k+1/k)$ and $D(k+1/k)$ are obtained via the Agee - Turner matrix factorization algorithm given in Appendix A and described by Bierman in [2]. Thus, after a time update, we have the predicted state estimate computed from (5) and the U-D factors $U(k+1/k)$ and $D(k+1/k)$ such that $P(k+1/k) = U(k+1/k)\ D(k+1/k)\ U^T(k+1/k)$. If $U(k/k)$ and $D(k/k)$ are the U-D factors of $P(k/k)$, then the U-D factors of the product $\Phi(\Delta_k)P(k/k)\Phi^T(\Delta_k)$ in (6) are (since $\Phi(\Delta_k)$ is upper triangular),

$$U = \Phi(\Delta_k)U(k/k) \tag{9}$$

and

$$D = D(k/k) \tag{10}$$

The U-D factors of $P(k+1/k)$ are obtained via the algorithm of Appendix A using the U-D factors given in (9) and (10).

At a measurement update the state estimate $\hat{s}(k+1/k+1)$ is computed from (8). The U-D factors, $U(k+1/k+1)$ and $D(k+1/k+1)$, of $P(k+1/k+1)$ are computed by noting that $P(k+1/k+1)$ given by (7) is the sum of a positive definite matrix $P(k+1/k)$, whose U-D factors are available, and a symmetric dyad $c \cdot uu^T$ where $c = -(HP(k+1/k)H^T + R(t_k))^{-1}$ is a scalar and $U = P(k+1/k)H^T$ is a 3-vector. The algorithm of Appendix A could be used to compute $U(k+1/k+1)$ and $D(k+1/k+1)$. However, the negative value of c presents some potential problems in using this algorithm so that we will use the algorithm of Appendix B to obtain the U-D factors of $P(k+1/k+1)$. The algorithm of Appendix B exploits the special structure of $c \cdot uu^T$ to compute the updated U-D factors more accurately than the algorithm of Appendix A.

INITIALIZATION OF THE KALMAN FILTER

At the beginning of the measurement sequence and at the end of large time breaks in the measurement sequence initial values of the position, velocity, and acceleration states and the U-D factors of their covariance matrix must be supplied to the Kalman filter. The quantities required for initialization of the Kalman filter are obtained by least squares estimation over the first N_s data points. Thus, if t_1 is the time of initialization, the estimate $\hat{s}(1/1)$

used to initialize the filter is obtained by minimizing,

$$\sum_{i=1}^{N_s} \left(m(t_i) - H\Phi(t_i - t_1)s(t_1)\right)^2 \tag{11}$$

with respect to $s(t_1)$. The least squares estimate is given by,

$$\hat{s}(1/1) = P \sum_{i=1}^{N_s} \Phi(t_i - t_1)H^T m(t_i), \tag{12}$$

where P is given by,

$$P = \sum_{i=1}^{N_s} \Phi(t_i - t_1)H^T H\Phi(t_i - t_1) \tag{13}$$

The covariance matrix of $\hat{s}(1/1)$ is calculated from,

$P(1/1) = P\hat{\sigma}^2$,

where $\hat{\sigma}^2$ is obtained from the least squares residuals as,

$$\hat{\sigma}^2 = \frac{1}{N_s - 3} \sum_{i=1}^{N_s} (m(t_i) - H\Phi(t_i - t_1)\hat{s}(1/1))^2 \tag{14}$$

The U-D factors of P(1/1) are obtained via the Cholesky decomposition algorithm given in Appendix C.

At the present time N_s is target dependent as in Table 1. In Table 1 T is the sampling period of the measurements.

TABLE 1

TARGET CATEGORY	N_s
Aircraft - no maneuver	max (20,1.5/T)
Aircraft - maneuver	max (20,1/T)
Ground Target	max (20,2/T)
Low Altitude - no maneuver	max (20,2/T)
Low Altitude - maneuver	max (20,1/T)
Helicopter	max (20,1.5/T)
Low Acceleration missile	max (20,1.5/T)
High Acceleration missile	max (20,1/T)

Also, $N_s \leq 50$.

MEASUREMENT NOISE COVARIANCE

The measurement noise variance $R(t_k)$ used in the Kalman filter is computed from past predicted filter residuals. Let $\hat{s}(k-p+2/k-p+1)$, $p = 1,P$ be the P predicted states preceding t_K. $R(t_K)$ is computed from

$$R(t_K) = \frac{1}{P} \sum_{p=1}^{P} (m(t_{k-p+2}) - H\hat{s}(k-p+2/k-p+1))^2 \tag{16}$$

Presently, P = 20.

STATE NOISE COVARIANCE

The state noise covariance for the Kalman filter should measure the uncertainty about the assumption of constant acceleration. Thus, it is realistic to choose $\sqrt{q}$ to measure the rate of change of acceleration. We could do this by differencing the filtered acceleration, $\hat{s}_3(k/k)$. However, to obtain a quicker response, we have chosen to estimate a smoothed acceleration, $\hat{s}_3^q(k)$ and difference this quantity. Let $\hat{s}^q(k)$ be a 3-vector of position, velocity, and acceleration and be the value of $s(t_k)$ which minimizes

$$(s(t_k)-\hat{s}(k/k))^T P^{-1}(k/k)(s(t_k)-\hat{s}(k/k))+\overline{R}_k^{-1}\sum_{i=1}^{N_q}(m(t_{k+i})-H\Phi(t_{k+i}-t_k)s(t_k)), \quad (17)$$

where $\overline{R}_k$ is an average measurement noise variance over the interval (t_k, t_{k+Nq}). Minimizing (17) with respect to $s(t_k)$ results in the smoother equation

$$\hat{s}^q(k)=\hat{s}(k/k)+P^q(k)\overline{R}_k^{-1}\sum_{i=1}^{N_q}\Phi^T(t_{k+i}-t_k)H^T(m(t_{k+i})-H\Phi(t_{k+i}-t_k)\hat{s}(k/k)) \quad (18)$$

In (18) $P^q(k)$ is the covariance matrix of $\hat{s}^q(k)$,

$$P^{q^{-1}}(k)=P^{-1}(k/k)+\overline{R}_k^{-1}\sum_{i=1}^{N_q}\Phi^T(t_{k+i}-t_k)H^T H\Phi(t_{k+i}-t_k) \quad (19)$$

The computation of $\overline{R}_k$ is described in Appendix D.

The rate of change of acceleration used in estimating q is computed by differencing the smoothed acceleration $\hat{s}_3^q(k)$. Thus, we compute

$$\dot{a}(k) = \frac{\hat{s}_3^q(k)-\hat{s}_3^q(k-1)}{t_k-t_{k-1}} \quad (20)$$

Rather than use $\dot{a}(k) = \sqrt{q(k)}$, it is much better to smooth the sequence, $\dot{a}(k)$, with a fading memory filter and use the smooth filter output to compute $\sqrt{q(k)}$. Thus, we estimate $\bar{\dot{a}}(k)$ to minimize

$$\sum_{i=0}^{k-2} W^i(\dot{a}(k-i)-\bar{\dot{a}}(k))^2 \quad (21)$$

where $0<w\leq 1$. The resulting steady state filter equation for $\bar{\dot{a}}(k)$, obtained by minimizing (21) and then finding the limiting steady state result is

$$\bar{a}(k+1) = \bar{a}(k) + (1-W)(\dot{a}(k+1)-\bar{a}(k)) \tag{22}$$

Then $\sqrt{q(k)} = \bar{a}(k)$.

The weight W used in (22) is target dependent. The weights for various categories of targets are given in Table 2. The values of W in this table have not been optimized in any way but have yielded good performance on data from the various target types. These values may be updated on the basis of future experience. An absolute upper limit, Q_u, which is also target dependent, has been placed on $\sqrt{q(k)}$. This limit, while seldom reached, is necessary to prevent the Kalman filter from responding violently to occasional bad data (not spikes) situations. The values of Q_u are also given in Table 2. The values of W given table 2 are for a sample rate of 20/sec. The values of W to be used for other sample rates are calculated in the program from the table values in the following way. Let W_R be the table value (20/s value) for a given target type and let W be the value to be used at the actual sampling period, T. Then

$$\log_{10} W = \left(\frac{T}{.05}\right) \log_{10} W_R \tag{23}$$

TABLE 2

TARGET CATEGORY	1-W x	1-W y	1-W z	Q_u x	Q_u y	Q_u z
Aircraft - no maneuver	.05	.05	.05	50	50	50
Aircraft - maneuver	.13	.13	.13	500	500	500
Ground Target	.05	.05	.05	20	20	20
Low Altitude - no maneuver	.05	.05	.05	50	50	50
Low Altitude - maneuver	.13	.13	.05	200	200	50
Helicopter	.05	.05	.05	50	50	50
Low Acceleration missile	.13	.13	.13	200	200	200
High Acceleration missile	.25	.25	.25	5000	5000	5000

Presently, the look interval for state noise adaption is given by N_q = max (20,1.5/T), where T is sampling interval of the measurements. Also, $N_q \leq 50$.

THE ADJOINT FILTER

The adjoint filter is a Kalman like filter running backwards in time which uses the Kalman filter residuals as input and also uses the Kalman filter gain vector. The optimal fixed interval smoother estimates are obtained by properly combining the Kalman filter states with the adjoint filter states. Let $\hat{\lambda}(k-1/k)$ and $\hat{\lambda}(k/k)$ be the predicted and filtered state vectors (3-vectors) of the adjoint filter. Also, let $\Lambda(k-1/k)$ and $\Lambda(k/k)$ be the covariance matrices of the predicted and filtered adjoint state vectors. The equations governing the adjoint state vectors and their covariances are given by:

$$\hat{\lambda}(k/k+1) = \Phi^T(\Delta_k)\hat{\lambda}(k+1/k+1), \tag{24}$$

$$\Lambda(k/k+1) = \Phi^T(\Delta_k)\Lambda(k+1/k+1)\Phi(\Delta_k), \tag{25}$$

$$\hat{\lambda}(k/k) = \hat{\lambda}(k/k+1) - H^T D_k^{-1}(r(k/k-1)+D_k K_k^T \hat{\lambda}(k/k+1)), \tag{26}$$

$$\Lambda(k/k) = (I-K_kH)^T\Lambda(k/k+1)(I-K_kH)+H^T D_j^{-1} H. \tag{27}$$

In the above equations $r(k/k-1)$ is the predicted Kalman filter residual,

$$r(k/k-1) = m(t_k)-H\hat{s}(k/k-1), \tag{28}$$

D_k is the covariance of this predicted residual,

$$D_k = HP(k/k-1)H^T+R(t_k), \tag{29}$$

and K_k is the Kalman filter gain vector,

$$K_k = P(k/k-1)H^T D_k^{-1} \tag{30}$$

In contrast to the Kalman filter the equations of the adjoint filter are implemented directly rather than using a matrix square root implementation.

The adjoint filter is initialized at the final trajectory time, t_N, and at the end of any measurement sequence preceeding a large time break. The adjoint filter is initialized by,

$$\hat{\lambda}(N/N) = -H^T D_N^{-1} r(N/N-1) \tag{31}$$

$$\Lambda(N/N) = H^T D_N^{-1} H \tag{32}$$

OPTIMAL SMOOTHED ESTIMATES

The optimal, fixed interval smoothed estimates are obtained by combining the states of the Kalman filter with the states of the adjoint filter. Assuming trajectory state estimates are desired only at the measurement times, the smoothed estimates are computed from,

$$\hat{s}(k) = \hat{s}(k/k) - P(k/k)\hat{\lambda}(k/k+1) \tag{33}$$

and the covariance of this estimate can be computed from (if desired for output),

$$P(k) = P(k/k) - P(k/k)\Lambda(k/k+1)P(k/k) \tag{34}$$

APPENDIX A

The matrix factorization algorithm described below for obtaining the U-D factors of a positive definite matrix plus a symmetric dyad was first reported in [3] and later described by Bierman in [2].

Given a positive definite matrix $\hat{P}$, a vector V, and a scalar c, we form the matrix $\tilde{P}$,

$$\tilde{P} = \hat{P} + cVV^T$$

Suppose we have a unit upper triangular matrix $\hat{U}$ and a positive diagonal matrix $\hat{D}$ such that $P = \hat{U}\hat{D}\hat{U}^T$. An algorithm for computing the U-D factors, $\tilde{U}$ and $\tilde{D}$, of $\tilde{P}$ is given by the following sequence of steps,

$$\tilde{D}(I) \doteq \hat{D}(I) + cV^2(I) \qquad I=N,1$$

$$\left.\begin{array}{l} \left.\begin{array}{l} V(K) \doteq V(K) - V(I)\,\hat{U}(K,I) \\ \tilde{U}(K,I) \doteq \hat{U}(K,I) + \dfrac{cV(I)}{\tilde{D}(I)}\;V(K) \end{array}\right\} K=1,I-1 \\ c \doteq \dfrac{c\hat{D}(I)}{\tilde{D}(I)} \end{array}\right\} I=N,2$$

In the above algorithm the symbol $\doteq$ represents the FORTRAN replacement equality. Note that the computation of the U-D factors can be done in place with the D(I) stored along the diagonal of U.

APPENDIX B

The following algorithm is taken from Bierman [2], p77. This algorithm computes the U-D factors of the posterior Kalman filter covariance matrix, $P(k+1/k+1) = \hat{P}$.

$$\hat{P} = \tilde{P} - \tilde{P}H^T(H\tilde{P}H^T + R)^{-1}H\tilde{P} \tag{B1}$$

Let $\tilde{U}$ and $\tilde{D}$ be the U-D factors of $\tilde{P}$,

$$\tilde{P} = \tilde{U}\tilde{D}\tilde{U}^T \tag{B2}$$

Let f, v, and g_J be N-vectors

$$f = \tilde{U}^T H^T$$

$$v = \tilde{D}f$$

$$\alpha_1 = v(1)f(1) + R, \quad g_2^T = (v(1)0---0)$$

$$\hat{D}(1) = \tilde{D}(1)R/\alpha_1$$

$$\left.\begin{array}{l} \alpha_J = \alpha_{J-1} + v(J)f(J) \\ \hat{D}(J) = \tilde{D}(J)\alpha_{J-1}/\alpha_J \\ \left.\begin{array}{l} \hat{U}(K,J) = \tilde{U}(K,J) - f(J)g_J(K)/\alpha_{J-1} \\ g_{J+1}(K) = g_J(K) + v(J)\tilde{U}(K,J) \end{array}\right\} K=1,J \end{array}\right\} J=2,N$$

The vector g_{N+1}/α_N will be the Kalman filter gain. The computation of U and D can be done in place, if desired.

APPENDIX C

The algorithm presented below obtains the U-D factors of the positive definite, NXN matrix P, $P = UDU^T$, where U is upper triangular and D is a positive diagonal matrix.

$$D(N) = P(N,N)$$

$$U(K/N) = P(K,N)/D(N), \quad K=1, N-1$$

$$D(I) = P(I,I) - \sum_{K=I+1}^{N} U^2(I,K)D(K), \quad I=N-1,1$$

$$U(K,I) = (P(K,I) - \sum_{L=K+1}^{N} U(K,L)D(L)U(I,L))/D(I), K=1, I-1, I=N-1,2$$

APPENDIX D

The average noise variance, $\overline{R}_K$, over the interval (t_k, t_{k+N_q}) is computed by fitting a quadratic curve to the measurements in the interval and then letting $\overline{R}_K$ be the variance of the measurement residuals from the fit. Consider the quadratic model,

$$m(t_{k+i}) = a + b(t_{k+i}-t_k) + c(t_{k+i}-t_k)^2, \; i=1, N_q$$

Let $\hat{a}$, $\hat{b}$, $\hat{c}$ be the least squares estimates of a, b, c. The residuals from this fit are

$$\delta(k+i) = m(t_{k+i}) - \hat{a} - \hat{b}\,(t_{k+i}-t_k) - \hat{c}\,(t_{k+i}-t_k)^2, \; i=1, N_q$$

The noise variance $\overline{R}_K$ used is computed by

$$\overline{R}_K = \frac{1}{N_q-3} \sum_{i=1}^{N_q} \delta^2(k+i)$$

This value is used only if $N_q \geq 10$. If $N_q < 10$, $\overline{R}_K = R(t_k)$ is used.

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